Circle Packings from Penrose Tilings

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To a triangulation of a surface, associate a collection of circles
To each vertex, associate a circle:
Circle Packings

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- An edge indicates a tangency:
Circle Packings

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- An edge indicates a tangency:

- A face indicates a triplet of tangencies:
Circle packings add geometric information to a combinatorial object.
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The nerve of the packing is the graph obtained by adding vertices and edges.
Thurston introduced circle packings in 1985. The theory gives a discrete version of complex analysis.
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There are many applications, including “brain flattening.”
An application: the Riemann Mapping Theorem

- When viewed on a small scale, a complex analytic function looks like a rotation composed with a dilation.
An application: the Riemann Mapping Theorem

When viewed on a small scale, a complex analytic function looks like a rotation composed with a dilation.

As such, small circles get carried into small circles.
Theorem

Given a simply connected, open, proper subset of the plane, there is a (essentially unique) complex analytic bijection to the unit disk.
To construct such a map, choose a positive number $r$ and construct the “penny-packing” of radius $r$ in the plane.

$r = 0.1$
Use the circles whose centers lie within the curve to define a triangulation.
Riemann Mapping Theorem

Pack this triangulation inside the unit disk.
Riemann Mapping Theorem

- Pack this triangulation inside the unit disk.

- Use this to define a function from the interior of the curve to the disk.
Repeat with increasingly small values of $r$. 

$r = 0.05$
Riemann Mapping Theorem

\[ r = 0.01 \]
Riemann Mapping Theorem

As $r \to 0$, these functions converge to a bijective complex analytic function.
Theorem

If appropriate boundary information is specified, a circle packing exists and is (essentially) unique.
There is an efficient algorithm for finding circle packings. The key is to notice that the angles around each interior vertex add to $2\pi$. 

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Circle Packings from Penrose Tilings
An algorithm

- Initialize all the radii on the boundary.
An algorithm

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- Assign arbitrary radii to the interior vertices.
With this information, we can calculate an “angle sum” at each interior vertex. Given a face in the triangulation, the Law of Cosines shows how to compute the angles formed by the triple tangency.
An algorithm

- With this information, we can calculate an “angle sum” at each interior vertex. Given a face in the triangulation, the Law of Cosines shows how to compute the angles formed by the triple tangency.

- At an interior vertex $v$, we can then compute the “angle sum” $\theta_s(v)$.
We then adjust the radius to make the angle sum equal $2\pi$. 
An algorithm

- We then adjust the radius to make the angle sum equal $2\pi$.

- If we iterate over all interior vertices, the radii converge to a circle packing.
No boundary? No problem.

- Suppose we have a triangulation \( K \) of the plane.
Maximal packings

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- Suppose we have a triangulation $K$ of the plane.
- Write $K$ as an increasing union of finite triangulations $K = \bigcup_i K_i$
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- Suppose we have a triangulation \( K \) of the plane.
- Write \( K \) as an increasing union of finite triangulations \( K = \bigcup_i K_i \)
- Pack each \( K_i \) in hyperbolic space and study the limit.
Maximal packings: an example
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Maximal packings: an example

- The packing obtained is called the **maximal packing**.
- In this example, the nerve of the packing is congruent to the original tiling.
- We asked what would happen if we looked at packings that come from Penrose tilings.
Most of the tilings that we first think of have translational symmetry:
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Penrose (and others) found collections of tiles that create tilings with no translational symmetries.
Penrose tilings

From *Tilings and Patterns*, Grünbaum & Shephard
A later set of tiles consisted of two rhombs:
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Again, the tilings created have no translational symmetry:
Since Penrose tilings have no translational symmetry, they may appear to have no structure.
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However, they do! To explain this structure, we’ll break the rhombs up into half-rhombs.
With some experimentation, we find that all half-rhombs must uniquely fit into one of two groups:
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This creates a new tiling $T_1$: 

![Graphic representation of the tiling $T_1$]
That creates a new tiling $T_2$:
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that creates a new tiling $T_3$
That creates a new tiling $T_2$:

that creates a new tiling $T_3$

that creates a new tiling $T_4$
That creates a new tiling $T_2$:

that creates a new tiling $T_3$
that creates a new tiling $T_4$
that creates a new tiling...
We wondered if the geometry of the maximal packing from a Penrose tiling reflected the geometry of the original tiling.
It appeared not.
But we did notice a pattern after coloring by vertex type:
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and filling in:
The geometry of the inflated tilings seem to appear
As $n \to \infty$, the nerve corresponding to $T_n$, the $n^{th}$ tiling in the inflation hierarchy, becomes congruent to $T_n$. 
Consider triangulations given by vertices that fit into a single rhomb higher in the inflation hierarchy. We created circle packings by fixing the angles on the boundary.
Strategy of proof

In the same way that we glue together rhombs, we would like to glue together the corresponding packings.
Strategy of proof

- Since the radii do not match up exactly, there is some error in the angle sums along the seam.
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- Since $\sum_v \theta_s(v) = \pi \cdot (# \text{ of faces})$, we know that the sum of the angle excesses is the same as the sum of the angle deficits.
Applying the packing algorithm, we are able to study how the error in the angle sum changes after each step in the algorithm.
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As a result we find that, after running the algorithm, the radii on the boundaries has changed very little.

This means that we can continue the gluing process. At each step, the radii on the boundaries change by a smaller and smaller amount.
The geometry of the packings from Penrose tilings reflects the inflation hierarchy.
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- The gluing process we found seems to be novel and may be useful in other circle packing problems.
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- The gluing process we found seems to be novel and may be useful in other circle packing problems.
- We have constructed the maximal packing without using the limit of hyperbolic packings.